Comments on the D-instanton calculus in $(p, p+1)$ minimal string theory

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abstract

The FZZT and ZZ branes in $(p, p+1)$ minimal string theory are studied in terms of continuum loop equations. We show that systems in the presence of ZZ branes (D-instantons) can be easily investigated within the framework of the continuum string field theory developed by Yahikozawa and one of the present authors [1]. We explicitly calculate the partition function of a single ZZ brane for arbitrary p. We also show that the annulus amplitudes of ZZ branes are correctly reproduced.

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1 Introduction

Since conformally invariant boundary states were constructed in the worldsheet description [2, 3, 4], renewed interest in noncritical string theory has arisen [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]. These boundary states lead to two types of branes. One is of FZZT branes given by Neumann-like boundary states. They are extended in the weak coupling region of Liouville coordinate and naturally correspond to unmarked macroscopic-loop operators [18] in matrix models. The other is of ZZ branes given by Dirichlet-like boundary states. They are localized in the strong coupling region of Liouville coordinate and are shown in [5, 6, 7] to correspond to "eigenvalue instantons" in matrix models [19]. In particular, the decay amplitude of ZZ branes is identified with that of eigenvalues rolling from a local maximum of matrix-model potential, and the ZZ branes are identified with D-instantons in noncritical string theory [20, 21, 22].[1]

In this paper, we show that these FZZT and ZZ branes can be naturally understood within the framework of a continuum string field theory for macroscopic loops [1, 23]. The Fock space of this string field theory is realized by p pairs of free chiral fermions $c_a(\zeta)$ and $\bar{c}_a(\zeta)$ ($a = 0, 1, \cdots, p-1$), living on the complex plane whose coordinate is given by the boundary cosmological constant ζ. It is shown in [1] that their diagonal bilinears $\bar{c}_a(\zeta)c_a(\zeta)$ (bosonized as $\partial\varphi_a(\zeta)$) can be identified with marked macroscopic loops,

$$\partial\varphi_a(\zeta) = :\bar{c}_a(\zeta)c_a(\zeta): = \int_0^\infty dl\, e^{-\zeta l}\, \Psi(l) \tag{1.1}$$

with $\Psi(l)$ being the operator creating the boundary of length l. This implies that the unmarked macroscopic loops (FZZT branes) are described by

$$\varphi_a(\zeta) \sim \int_0^\infty \frac{dl}{l}\, e^{-\zeta l}\, \Psi(l). \tag{1.2}$$

Furthermore, it is shown in [1] that their off-diagonal bilinears $\bar{c}_a(\zeta)c_b(\zeta)$ ($a \neq b$) (bosonized as $e^{\varphi_a(\zeta)-\varphi_b(\zeta)}$) can be identified with the operator creating a soliton at the "spacetime coordinate" ζ [22]. In order for this operator to be consistent with the continuum loop equations (or the $W_{1+\infty}$ constraints) [24, 25, 26, 27, 28], the position of the soliton must be integrated as

$$D_{ab} \equiv \int \frac{d\zeta}{2\pi i}\, \bar{c}_a(\zeta)c_b(\zeta) = \int \frac{d\zeta}{2\pi i}\, e^{\varphi_a(\zeta)-\varphi_b(\zeta)}. \tag{1.3}$$

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[1] In what follows, the terms "ZZ branes" and "D-instantons" will be used interchangeably.

This integral can be regarded as defining an effective theory for the position of the soliton. In the weak coupling limit $g \to 0$ (g: the string coupling constant) the expectation value of $\bar{c}_a(\zeta)c_b(\zeta)$ behaves as $\exp\big(g^{-1}\Gamma_{ab}(\zeta) + O(g^0)\big)$, where the "effective action" Γ_{ab} is expressed as the difference of the disk amplitudes:

$$\Gamma_{ab} = \langle \varphi_a \rangle^{(0)} - \langle \varphi_b \rangle^{(0)} . \tag{1.4}$$

Thus, in the weak coupling limit, the soliton will get localized at a saddle point of Γ_{ab} and behave as a D-instanton (a ZZ brane). The relation (1.4) evaluated at the saddle point gives a well-known relation between disk amplitudes of FZZT and of ZZ branes [6]. One shall be able to extend this relation to arbitrarily higher-order amplitudes.

The main aim of the present paper is to elaborate the calculation around the saddle point performed in [1], and to establish the above relationship with a catalog of possible quantum numbers. We show that the partition functions of D-instantons for generic $(p, p+1)$ minimal string theories can be calculated explicitly, and also demonstrate that many of the known results obtained in Liouville field theory [6, 11, 15] and/or in matrix models [19, 13, 14] can be reproduced easily.

This paper is organized as follows. In section 2 we give a brief review on the continuum string field theory for macroscopic loops [1, 23]. In section 3 we evaluate one-instanton partition function for $(p, p+1)$ minimal string theory, and show that it correctly reproduces the value obtained in [19] (see also [13, 14]) for the pure gravity case ($p = 2$). In section 4 we make a detailed comparison of our analysis with that performed in Liouville field theory. In section 5 we calculate the annulus amplitudes of ZZ branes, and show that it reproduces the values obtained in [6, 15]. Section 6 is devoted to conclusion and discussions. In the present paper, we exclusively consider the unitary case $(p, q) = (p, p + 1)$. General (p, q) minimal strings can also be treated in a similar manner, and detailed analysis will be reported in the forthcoming paper.

2 Review of the noncritial string field theory

From the viewpoint of noncritical strings, $(p, p + 1)$ minimal string theory describes two-dimensional gravity coupled to matters of minimal unitary conformal field theory ($c_{\text{matter}} = 1 - 6/p(p+1)$). There the basic physical operators are the macroscopic loop operator $\Psi(l)$ which creates a boundary of length l. It is often convenient to introduce their Laplace

transforms

$$\partial\varphi_0(\zeta) \equiv \int_0^\infty dl\, e^{-\zeta l}\, \Psi(l), \tag{2.1}$$

and ζ is called the boundary cosmological constant. The left-hand side is written with a derivative for later convenience. Note that ζ can take a different value on each boundary. The connected correlation functions of macroscopic loop operators

$$\left\langle \partial\varphi_0(\zeta_1) \cdots \partial\varphi_0(\zeta_n) \right\rangle_{\mathrm{c}} \tag{2.2}$$

are expanded with respect to the string coupling constant $g\ (>0)$ as

$$\left\langle \partial\varphi_0(\zeta_1) \cdots \partial\varphi_0(\zeta_n) \right\rangle_{\mathrm{c}} = \sum_{h \geq 0} g^{-2+2h+n} \left\langle \partial\varphi_0(\zeta_1) \cdots \partial\varphi_0(\zeta_n) \right\rangle_{\mathrm{c}}^{(h)}, \tag{2.3}$$

where h is the number of handles. Note that $\left\langle \partial\varphi_0(\zeta) \right\rangle^{(0)}$ and $\left\langle \partial\varphi_0(\zeta_1)\, \partial\varphi_0(\zeta_2) \right\rangle_{\mathrm{c}}^{(0)}$ correspond to the disk and annulus (cylinder) amplitudes, respectively.

The operator formalism of the continuum string field theory [1] is constructed with the following steps:

STEP1:

We introduce p pairs of free chiral fermions $c_a(\zeta)$ and $\bar{c}_a(\zeta)$ ($a = 0, 1, \cdots, p-1$) which have the OPE

$$c_a(\zeta)\, \bar{c}_b(\zeta') \sim \frac{\delta_{ab}}{\zeta - \zeta'} \sim \bar{c}_a(\zeta)\, c_b(\zeta'). \tag{2.4}$$

This can be bosonized with p free chiral bosons $\varphi_a(\zeta)$ (defined with the OPE: $\varphi_a(\zeta)\, \varphi_b(\zeta') \sim + \delta_{ab} \ln(\zeta - \zeta')$) as

$$\bar{c}_a(\zeta) = K_a : e^{\varphi_a(\zeta)} :, \quad c_a(\zeta) = K_a : e^{-\varphi_a(\zeta)} :, \tag{2.5}$$

where K_a are cocycles which ensure the correct anticommutation relations between fermions with different indices $a \neq b$.[2] In this paper the normal ordering $:\ :$ are always taken so as to respect the $\mathrm{SL}(2, \mathbb{C})$-invariant vacuum $|0\rangle$. The chiral field $\varphi_a(\zeta)$ can in turn be expressed as

$$\partial\varphi_a(\zeta) = :\bar{c}_a(\zeta)\, c_a(\zeta): . \tag{2.6}$$

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[2]In [1] K_a are chosen to be $K_a \equiv \prod_{b=0}^{a-1}(-1)^{p_b}$ with p_a being the fermion number of the a-th species.

STEP2:

We impose the $\mathbb{Z}_p$-twisted boundary conditions on the fermions as

$$c_a(e^{2\pi i}\zeta) = c_{[a+1]}(\zeta), \quad \bar{c}_a(e^{2\pi i}\zeta) = \bar{c}_{[a+1]}(\zeta) \quad ([a] \equiv a \;(\mathrm{mod}\; p)), \tag{2.7}$$

and correspondingly,

$$\partial\varphi_a(e^{2\pi i}\zeta) = \partial\varphi_{[a+1]}(\zeta). \tag{2.8}$$

This can be realized by inserting the $\mathbb{Z}_p$-twist field $\sigma(\zeta)$ at $\zeta = 0$ and at $\zeta = \infty$, with which the chiral field $\partial\varphi_a(\zeta)$ is expanded as

$$\langle\sigma|\cdots\partial\varphi_a(\zeta)\cdots|\sigma\rangle = \langle\sigma|\cdots\frac{1}{p}\sum_{n\in\mathbb{Z}}\omega^{-na}\alpha_n\,\zeta^{-n/p-1}\cdots|\sigma\rangle, \quad \left(\omega \equiv e^{2\pi i/p}\right) \tag{2.9}$$

$$\left[\alpha_n,\,\alpha_m\right] = n\,\delta_{n+m,0}. \tag{2.10}$$

Here $\langle\sigma| \equiv \langle 0|\sigma(\infty)$ and $|\sigma\rangle \equiv \sigma(0)|0\rangle$.

STEP3:

We introduce the generators of the $W_{1+\infty}$ algebra [29], $\{W_n^k\}$ ($k = 1, 2, \cdots; n \in \mathbb{Z}$), that are given by the mode expansion of the currents

$$W^k(\zeta) \equiv \sum_{n\in\mathbb{Z}}W_n^k\,\zeta^{-n-k} = \sum_{a=0}^{p-1}:\bar{c}_a(\zeta)\,\partial_\zeta^{k-1}c_a(\zeta): \quad (k = 1, 2, \cdots). \tag{2.11}$$

Finally we introduce the state $|\Phi\rangle$ that satisfies the vacuum condition of the $W_{1+\infty}$ constraints:

$$W_n^k\,|\Phi\rangle = 0 \quad (k \geq 1,\; n \geq -k+1). \tag{2.12}$$

This constraint is shown to be equivalent to the continuum loop equations [24, 25, 30, 26, 27, 28]. In addition to the $W_{1+\infty}$ constraints, we further require that $|\Phi\rangle$ be a decomposable state.[3]

STEP4:

One can prove that the connected correlation functions of macroscopic loop operators are

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[3]A state $|\Phi\rangle$ is said to be decomposable if it can be written as $|\Phi\rangle = e^H|\sigma\rangle$, where H is a bilinear form of the fermions, $H = \int d\zeta\,d\zeta'\sum_{a,b}\bar{c}_a(\zeta)\,h_{ab}(\zeta,\zeta')\,c_b(\zeta')$. This is equivalent to the statement that $\tau(x) = \langle\sigma|\exp\{\sum_{n=1}^{\infty}x_n\alpha_n\}|\Phi\rangle$ is a τ function of the p-th reduced KP hierarchy [31]. It is proved in [27, 28] that this set of conditions ($W_{1+\infty}$ constraints and decomposability) is equivalent to the Douglas equation, $[P, Q] = 1$ [32].

given as cumulants (or connected parts) of the following correlation functions [1]:[4]

$$\left\langle \partial\varphi_0(\zeta_1)\cdots\partial\varphi_0(\zeta_n)\right\rangle \equiv \frac{\left\langle -\frac{B}{g}\middle| :\partial\varphi_0(\zeta_1)\cdots\partial\varphi_0(\zeta_n): \middle| \Phi\right\rangle}{\left\langle -\frac{B}{g}\middle| \Phi\right\rangle}. \tag{2.13}$$

Here the state

$$\left\langle -\frac{B}{g}\middle| \equiv \left\langle \sigma\middle| \exp\left(-\frac{1}{g}\sum_{n=1}^{\infty} B_n\alpha_n\right) \tag{2.14}$$

characterizes the theory, and the (p,q) minimal string is realized by taking $B_{p+q} \neq 0$ and $B_n = 0$ ($\forall n \geq p+q+1$).

From the viewpoint of two-dimensional gravity, α_1 creates the lowest-dimensional operator on random surfaces, so that B_1 ($\equiv \mu$) should correspond to the bulk cosmological constant in the unitary minimal strings. In fact, if we choose $B_1 = \mu$, $B_{2p+1} = -4p/(p+1)(2p+1)$ and $B_n = 0$ ($n \neq 0$ or $2p+1$), then we obtain the following disk and annulus amplitudes for marked macroscopic loops [1]:

$$\left\langle \partial\varphi_0(\zeta)\right\rangle^{(0)} = \frac{2^{(p-1)/p}}{p+1}\left[\left(\zeta + \sqrt{\zeta^2 - \mu}\right)^{(p+1)/p} + \left(\zeta - \sqrt{\zeta^2 - \mu}\right)^{(p+1)/p}\right], \tag{2.15}$$

$$\left\langle \partial\varphi_0(\zeta_1)\,\partial\varphi_0(\zeta_2)\right\rangle_{\rm c}^{(0)} = \partial_{\zeta_1}\partial_{\zeta_2}\Bigg[\ln\Bigg\{\left(\zeta_1 + \sqrt{\zeta_1^2 - \mu}\right)^{1/p} + \left(\zeta_1 - \sqrt{\zeta_1^2 - \mu}\right)^{1/p}$$
$$- \left(\zeta_2 + \sqrt{\zeta_2^2 - \mu}\right)^{1/p} - \left(\zeta_2 - \sqrt{\zeta_2^2 - \mu}\right)^{1/p}\Bigg\}$$
$$- \ln\left(\zeta_1 - \zeta_2\right)\Bigg], \tag{2.16}$$

which agree with the matrix model results [18]. The amplitudes of the corresponding FZZT branes are obtained by integrating the above amplitudes. In particular, the annulus amplitudes of the FZZT branes are given by

$$\left\langle \varphi_0(\zeta_1)\,\varphi_0(\zeta_2)\right\rangle_{\rm c}^{(0)} = \ln\Bigg\{\left(\zeta_1 + \sqrt{\zeta_1^2 - \mu}\right)^{1/p} + \left(\zeta_1 - \sqrt{\zeta_1^2 - \mu}\right)^{1/p}$$
$$- \left(\zeta_2 + \sqrt{\zeta_2^2 - \mu}\right)^{1/p} - \left(\zeta_2 - \sqrt{\zeta_2^2 - \mu}\right)^{1/p}\Bigg\}$$
$$- \ln\left(\zeta_1 - \zeta_2\right), \tag{2.17}$$

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[4]Since the normal ordering differs from that for the twisted vacuum $|\sigma\rangle$, the two-point function acquires a finite renormalizaion. This turns out to be the so-called nonuniversal term in the annulus amplitude [1]. The representation of loop correlators with free twisted bosons can also be found in [33], where open-closed string coupling is investigated.

with nonuniversal additive constants.

Moreover, by combining the $W_{1+\infty}$ constraint with the KP equation, one can easily show that the two-point function of cosmological term $u(\mu, g) \equiv \left(-g \frac{\partial}{\partial \mu}\right)^2 \ln \langle -B/g \, | \Phi \rangle$ satisfies the Painlevé-type equations [34, 35][5]

$$(p=2) \quad 4u^2 + \frac{2g^2}{3} \partial_\mu^2 u = \mu, \tag{2.18}$$

$$(p=3) \quad 4u^3 + \frac{3g^2}{2} (\partial_\mu u)^2 + 3g^2 u \, \partial_\mu^2 u + \frac{g^4}{6} \partial_\mu^4 u = -\mu, \tag{2.19}$$

$$\vdots$$

As for the solutions with soliton backgrounds, the crucial observation made in [1] is that the commutators between the $W_{1+\infty}$ generators and $\bar{c}_a(\zeta)c_b(\zeta)$ $(a \neq b)$ give total derivatives:

$$\left[W_n^k, \, \bar{c}_a(\zeta)c_b(\zeta) \right] = \partial_\zeta(*), \tag{2.20}$$

and thus the operator

$$D_{ab} \equiv \oint \frac{d\zeta}{2\pi i} \, \bar{c}_a(\zeta)c_b(\zeta) \tag{2.21}$$

commutes with the $W_{1+\infty}$ generators:

$$\left[W_n^k, \, D_{ab} \right] = 0. \tag{2.22}$$

Here the contour integral in (2.21) needs to surround the point of infinity $(\zeta = \infty)$ p times in order to resolve the $\mathbb{Z}_p$ monodromy. Equation (2.22) implies that if $|\Phi\rangle$ is a solution of the $W_{1+\infty}$ constraints (2.12), then so is $D_{a_1 b_1} \cdots D_{a_r b_r} |\Phi\rangle$. We will see that the latter can actually be identified with an r-instanton solution.[6]

By using the weak field expansions, the expectation value of D_{ab} can be expressed as

$$\begin{aligned}
\langle D_{ab} \rangle &= \oint \frac{d\zeta}{2\pi i} \left\langle e^{\varphi_a(\zeta) - \varphi_b(\zeta)} \right\rangle \\
&= \oint \frac{d\zeta}{2\pi i} \exp \left\{ \left\langle e^{\varphi_a(\zeta) - \varphi_b(\zeta)} - 1 \right\rangle_c \right\} \\
&= \oint \frac{d\zeta}{2\pi i} \exp \left\{ \langle \varphi_a(\zeta) - \varphi_b(\zeta) \rangle + \frac{1}{2} \langle (\varphi_a(\zeta) - \varphi_b(\zeta))^2 \rangle_c + \cdots \right\}.
\end{aligned} \tag{2.23}$$

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[5]If one takes account of the doubling in matrix models with even potentials, the two-point function will be replaced by $f \equiv 2u$.

[6]Note that if the decomposability condition is further imposed, the only possible form for the collection of instanton solutions should be $|\Phi, \theta\rangle \equiv \prod_{a \neq b} \exp(\theta_{ab} D_{ab}) |\Phi\rangle$ with fugacity θ_{ab} [23].

Since connected n-point functions have the following expansion in g:

$$\left\langle \partial \varphi_{a_1}(\zeta_1) \cdots \partial \varphi_{a_n}(\zeta_n) \right\rangle_{\rm c} = \sum_{h=0}^{\infty} g^{-2+2h+n} \left\langle \partial \varphi_{a_1}(\zeta_1) \cdots \partial \varphi_{a_n}(\zeta_n) \right\rangle_{\rm c}^{(h)}, \qquad (2.24)$$

leading contributions to the exponent of (2.23) in the weak coupling limit come from spherical topology ($h = 0$):

$$\langle D_{ab} \rangle = \oint \frac{d\zeta}{2\pi i} \, e^{(1/g)\, \Gamma_{ab}(\zeta) + (1/2)\, K_{ab}(\zeta) + O(g)} \qquad (2.25)$$

with

$$\Gamma_{ab}(\zeta) \equiv \left\langle \varphi_a(\zeta) \right\rangle^{(0)} - \left\langle \varphi_b(\zeta) \right\rangle^{(0)}, \qquad K_{ab}(\zeta) \equiv \left\langle \left(\varphi_a(\zeta) - \varphi_b(\zeta) \right)^2 \right\rangle_{\rm c}^{(0)}. \qquad (2.26)$$

Thus, in the weak coupling limit the integration is dominated by the value around a saddle point in the complex ζ plane.

3 Saddle point analysis

In this section, we make a detailed calculation of the integral (2.25) around a saddle point, up to $e^{O(g^1)}$.

The functions $\Gamma_{ab}(\zeta)$ and $K_{ab}(\zeta)$ can be calculated, basically by integrating the disk and annulus amplitudes ((2.15) and (2.16)), followed by analytic continuation $\zeta_1 \to e^{2\pi i a}\zeta_1$, $\zeta_2 \to e^{2\pi i b}\zeta_2$ and by taking the limit $\zeta_1, \zeta_2 \to \zeta$. For example, Γ_{ab} is calculated as

$$\begin{aligned} \Gamma_{ab}(\zeta) &= \left\langle \varphi_a(\zeta) \right\rangle^{(0)} - \left\langle \varphi_b(\zeta) \right\rangle^{(0)} \\ &= \left\langle \varphi_0(e^{2\pi i a}\zeta) \right\rangle^{(0)} - \left\langle \varphi_0(e^{2\pi i b}\zeta) \right\rangle^{(0)} \\ &= \int_{e^{2\pi i b}\zeta}^{e^{2\pi i a}\zeta} d\zeta' \left\langle \partial \varphi_0(\zeta') \right\rangle^{(0)}, \end{aligned} \qquad (3.1)$$

and we obtain

$$\Gamma_{ab}(\zeta) = \frac{p}{2^{1/p}(p+1)} \mu^{(2p+1)/2p} \left[\frac{1}{2p+1} \Big\{ (\omega^a - \omega^b)s^{(2p+1)/p} + (\omega^{-a} - \omega^{-b})s^{-(2p+1)/p} \Big\} \right. $$
$$\left. - \Big\{ (\omega^a - \omega^b)s^{1/p} + (\omega^{-a} - \omega^{-b})s^{-1/p} \Big\} \right], \qquad (3.2)$$

where $\omega \equiv e^{2\pi i/p}$, and $s = s(\zeta)$ is defined as

$$s = \frac{\zeta}{\sqrt{\mu}} + \sqrt{\left(\frac{\zeta}{\sqrt{\mu}}\right)^2 - 1}, \qquad s^{-1} = \frac{\zeta}{\sqrt{\mu}} - \sqrt{\left(\frac{\zeta}{\sqrt{\mu}}\right)^2 - 1}. \qquad (3.3)$$

On the other hand, the calculation of K_{ab} needs a special care because $\langle \varphi_a(\zeta_1)\varphi_b(\zeta_2)\rangle$ does not obey simple monodromy. This is due to the fact that the two-point function $\langle \varphi_a(\zeta_1)\varphi_b(\zeta_2)\rangle$ is defined with the normal ordering : : that respects the SL$(2,\mathbb{C})$ invariant vacuum:

$$\langle \varphi_a(\zeta_1)\varphi_b(\zeta_2)\rangle = \frac{\langle -B/g| :\varphi_a(\zeta_1)\varphi_b(\zeta_2): |\Phi\rangle}{\langle -B/g|\Phi\rangle}. \tag{3.4}$$

In fact, by using the definition $:\varphi_a(\zeta_1)\,\varphi_b(\zeta_2): = \varphi_a(\zeta_1)\,\varphi_b(\zeta_2) - \delta_{ab}\,\ln(\zeta_1 - \zeta_2)$, the two-point functions are expressed as

$$
\begin{aligned}
\langle \varphi_a(\zeta_1)\varphi_b(\zeta_2)\rangle &= \frac{\langle -B/g|\,\varphi_a(\zeta_1)\varphi_b(\zeta_2)\,|\Phi\rangle}{\langle -B/g|\Phi\rangle} - \delta_{ab}\ln(\zeta_1 - \zeta_2) \\
&= \frac{\langle -B/g|\,\varphi_0(e^{2\pi i a}\zeta_1)\varphi_0(e^{2\pi i b}\zeta_2)\,|\Phi\rangle}{\langle -B/g|\Phi\rangle} - \delta_{ab}\ln(\zeta_1 - \zeta_2) \\
&= \frac{\langle -B/g|\,:\varphi_0(e^{2\pi i a}\zeta_1)\varphi_0(e^{2\pi i b}\zeta_2):\,|\Phi\rangle}{\langle -B/g|\Phi\rangle} \\
&\quad + \ln(e^{2\pi i a}\zeta_1 - e^{2\pi i b}\zeta_2) - \delta_{ab}\ln(\zeta_1 - \zeta_2) \\
&= \langle \varphi_0(e^{2\pi i a}\zeta_1)\,\varphi_0(e^{2\pi i b}\zeta_2)\rangle + \ln(e^{2\pi i a}\zeta_1 - e^{2\pi i b}\zeta_2) - \delta_{ab}\ln(\zeta_1 - \zeta_2). \tag{3.5}
\end{aligned}
$$

We thus obtain[7]

$$
\begin{aligned}
\langle \varphi_a(\zeta_1)\varphi_b(\zeta_2)\rangle_{\mathrm{c}}^{(0)} = \ln\Big[&\omega^a\big(\zeta_1 + \sqrt{\zeta_1^2 - \mu}\big)^{1/p} + \omega^{-a}\big(\zeta_1 - \sqrt{\zeta_1^2 - \mu}\big)^{1/p} \\
&- \omega^b\big(\zeta_2 + \sqrt{\zeta_2^2 - \mu}\big)^{1/p} - \omega^{-b}\big(\zeta_2 - \sqrt{\zeta_2^2 - \mu}\big)^{1/p}\Big] \\
&- \delta_{ab}\ln(\zeta_1 - \zeta_2), \tag{3.6}
\end{aligned}
$$

from which $K_{ab}(\zeta)$ are calculated to be

$$
\begin{aligned}
K_{ab}(\zeta) = &\ln\Big(\omega^a\, s^{1/p} - \omega^{-a}\, s^{-1/p}\Big) + \ln\Big(\omega^b\, s^{1/p} - \omega^{-b}\, s^{-1/p}\Big) - 2\ln\big(s - s^{-1}\big) \\
&- \ln\Big[\big(\omega^a - \omega^b\big) s^{1/p} + \big(\omega^{-a} - \omega^{-b}\big) s^{-1/p}\Big] \\
&- \ln\Big[\big(\omega^b - \omega^a\big) s^{1/p} + \big(\omega^{-b} - \omega^{-a}\big) s^{-1/p}\Big] \\
&+ 2\ln\frac{2}{p\sqrt{\mu}}. \tag{3.7}
\end{aligned}
$$

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[7]Although these amplitudes may have nonuniversal additive corrections, they will be totally canceled in the calculation of Γ_{ab} and K_{ab}. In this sense, the value of the partition function of a D-instanton must be universal as in the matrix model cases [13].

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In order to make a further calculation in a well-defined manner, it is convenient to introduce new variables z and τ [11], for which the functions Γ_{ab} and K_{ab} are single-valued:

$$s = \frac{\zeta}{\sqrt{\mu}} + \sqrt{\left(\frac{\zeta}{\sqrt{\mu}}\right)^2 - 1} \equiv e^{ip\tau}, \qquad z \equiv \cos\tau = \frac{1}{2}(s^{1/p} + s^{-1/p}). \tag{3.8}$$

ζ is then expressed as

$$\frac{\zeta}{\sqrt{\mu}} = \cos p\tau = T_p(z), \tag{3.9}$$

where $T_n(z)$ $(n = 0, 1, 2, \cdots)$ are first Tchebycheff polynomials of degree n defined by $T_n(\cos\tau) \equiv \cos n\tau$. The monodromy of the operator $\phi_a(z) \equiv \varphi_a(\zeta)$ under twisted vacuum $|\sigma\rangle$ is expressed as

$$\phi_a(z)\,|\sigma\rangle = \phi_0(z_a)\,|\sigma\rangle \tag{3.10}$$

with $z_a \equiv \cos\tau_a \equiv \cos(\tau + 2\pi a/p)$.

We return to the calculation of the partition function in the presence of one soliton, $\langle D_{ab}\rangle$. Since the measure is written as $d\zeta = p\sqrt{\mu}\,U_{p-1}(z)\,dz$, we need to calculate the following integral:

$$\langle D_{ab}\rangle = \frac{p\sqrt{\mu}}{2\pi i} \int dz\,U_{p-1}(z)\,e^{-(1/g)\,\Gamma_{ab}(z)+(1/2)K_{ab}(z)+O(g)}. \tag{3.11}$$

Here $U_n(z)$ $(n = 0, 1, 2, \cdots)$ are second Tchebycheff polynomials of degree n defined by $U_n(\cos\tau) \equiv \sin(n+1)\tau/\sin\tau$. $T_n(z)$ and $U_n(z)$ are related as

$$T_n'(z) = n\,U_n(z), \qquad U_n'(z) = \frac{1}{1-z^2}\left[z\,U_n(z) - (n+1)\,T_{n+1}(z)\right]. \tag{3.12}$$

The function Γ_{ab} and their derivatives are easily obtained by using the formulas

$$\frac{dz_a}{dz} = \frac{\sin(\tau + 2\pi a/p)}{\sin\tau}, \qquad \frac{d^2 z_a}{dz^2} = \frac{\sin(2\pi a/p)}{\sin^3\tau}, \tag{3.13}$$

and are found to be

$$\Gamma_{ab}(z) = \langle\phi_a(z)\rangle^{(0)} - \langle\phi_b(z)\rangle^{(0)}$$
$$= \frac{2^{(p-1)/p}\,p}{p+1}\mu^{(2p+1)/2p}\left[\frac{1}{2p+1}\big(T_{2p+1}(z_a) - T_{2p+1}(z_b)\big) - (z_a - z_b)\right], \tag{3.14}$$

$$\Gamma_{ab}'(z) = -\frac{2^{(2p+1)/2p}\,p}{p+1}\mu^{(2p+1)/2p}(\omega^b - \omega^a)U_{p-1}(z)y^{-p-1}(y^{2(p+1)} - \omega^{-a-b}) \quad (y \equiv e^{i\tau}), \tag{3.15}$$

$$\Gamma_{ab}''(z) = \frac{z}{1-z^2}\,\Gamma_{ab}'(z)$$
$$- \frac{p\,\omega^{(p-1)/p}}{p+1}\mu^{(2p+1)/2p}\frac{1}{1-z^2}\left[(2p+1)\big(T_{2p+1}(z_a) - T_{2p+1}(z_b)\big) - (z_a - z_b)\right]. \tag{3.16}$$

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As for K_{ab}, one obtains the following formula from (3.6):

$$
\left\langle \left(\phi_a(z_1) - \phi_b(z_1) \right) \left(\phi_c(z_2) - \phi_d(z_2) \right) \right\rangle_c^{(0)}
$$
$$
= \ln \frac{(z_{1a} - z_{2c})(z_{1b} - z_{2d})}{(z_{1a} - z_{2d})(z_{1b} - z_{2c})} - (\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}) \ln \left[\sqrt{\mu} \left(T_p(z_1) - T_p(z_2) \right) \right], \qquad (3.17)
$$

and thus K_{ab} is expressed with z as

$$
K_{ab}(z) = \lim_{z_1, z_2 \to z} \left\langle \left(\phi_a(z_1) - \phi_b(z_1) \right) \left(\phi_a(z_2) - \phi_b(z_2) \right) \right\rangle_c^{(0)}
$$
$$
= \lim_{z_1, z_2 \to z} \ln \frac{(z_{1a} - z_{2a})(z_{1b} - z_{2b})}{(z_{1a} - z_{2b})(z_{1b} - z_{2a})(T_p(z_1) - T_p(z_2))^2 \mu}
$$
$$
= -\ln \left[-(z_a - z_b)^2 U_{p-1}(z_a) U_{p-1}(z_b) \right] - 2 \ln p \sqrt{\mu}. \qquad (3.18)
$$

The saddle points $z = z_* = \cos \tau_*$ are determined by solving the equation $\Gamma'_{ab}(z_*) = 0$, and are found to be[8]

$$
\tau_* = \left(-\frac{a+b}{p} + \frac{a+b+l}{p+1} \right) \pi, \quad l \in \mathbb{Z}. \qquad (3.19)
$$

They acquire the following changes under the transformation $z_* \to z_{*a} = \cos(\tau_* + 2\pi i a / p)$:

$$
z_{*a} = \cos \left(\frac{b-a}{p} - \frac{a+b+l}{p+1} \right) \pi = \cos \left(\frac{m}{p} - \frac{n}{p+1} \right) \pi, \qquad (3.20)
$$

$$
z_{*b} = \cos \left(\frac{b-a}{p} + \frac{a+b+l}{p+1} \right) \pi = \cos \left(\frac{m}{p} + \frac{n}{p+1} \right) \pi. \qquad (3.21)
$$

Here we have introduced two integers m and n as

$$
m \equiv b - a, \qquad n \equiv a + b + l. \qquad (3.22)
$$

It is easy to see that $T_{2p+1}(z)$ takes the following values at those shifted points:

$$
T_{2p+1}(z_{*a}) = \cos \left(\frac{m}{p} + \frac{n}{p+1} \right) \pi, \qquad (3.23)
$$

$$
T_{2p+1}(z_{*b}) = \cos \left(\frac{m}{p} - \frac{n}{p+1} \right) \pi. \qquad (3.24)
$$

From this, one easily obtains

$$
\Gamma_{ab}(z_*) = -\frac{8p}{2^{1/p}(2p+1)} \mu^{(2p+1)/2p} \sin \left(\frac{n}{p+1} \pi \right) \sin \left(\frac{m}{p} \pi \right), \qquad (3.25)
$$

---

[8]There are other possible saddle points determined by $U_{p-1}(z_*) = 0$. However, they only give irrelevant contributions to the integral because of the vanishing measure $d\zeta = p\sqrt{\mu} U_{p-1}(z) \, dz$ at such saddle points.

and

$$\Gamma''_{ab}(z_*) = \frac{8p}{2^{1/p} \sin^2 \tau_*} \mu^{(2p+1)/2p} \sin\left(\frac{n}{p+1}\pi\right) \sin\left(\frac{m}{p}\pi\right). \tag{3.26}$$

By using the relation $U_{p-1}(z_{*a}) = (\sin p\tau_* / \sin \tau_{*a}) U_{p-1}(z_*)$, $K_{ab}(z_*)$ can also be calculated easily, and is found to be

$$K_{ab}(z_*) = 2 \ln\left[\frac{\sqrt{\cos\left(\frac{2n\pi}{p+1}\right) - \cos\left(\frac{2m\pi}{p}\right)}}{2p\sqrt{2\mu} \sin \tau_* U_{p-1}(z_*) \sin\left(\frac{n\pi}{p+1}\right) \sin\left(\frac{m\pi}{p}\right)}\right]. \tag{3.27}$$

In order for the integration to give such nonperturbative effects that vanish in the limit $g \to +0$, we need to choose a contour along which $\mathrm{Re}\,\Gamma_{ab}(z)$ takes only negative values. In particular, (m, n) should be chosen such that $\Gamma_{ab}(z_*)$ is negative. This in turn implies that $\Gamma''_{ab}(z_*)$ is positive, and thus the corresponding steepest descent path intersects the saddle point in the pure-imaginary direction in the complex z plane. We thus take $z = z_* + it$ around the saddle point, so that the Gaussian integral becomes

$$\langle D_{ab}\rangle = \frac{p\sqrt{\mu}}{2\pi} U_{p-1}(z_*)\, e^{(1/2)K_{ab}(z_*)}\, e^{(1/g)\Gamma_{ab}(z_*)} \int_{-\infty}^{\infty} dt\, e^{-(1/2g)\Gamma''_{ab}(z_*)\, t^2}$$

$$= p\sqrt{\frac{\mu g}{2\pi}} \frac{U_{p-1}(z_*)}{\sqrt{\Gamma''_{ab}(z_*)}}\, e^{(1/2)K_{ab}(z_*)}\, e^{(1/g)\Gamma_{ab}(z_*)}. \tag{3.28}$$

Substituting all the values obtained above, we finally get

$$\langle D_{ab}\rangle = \frac{2^{1/2p}}{8\sqrt{2\pi p}} \sqrt{g}\, \mu^{-(2p+1)/4p} \sqrt{\frac{\cos\left(\frac{2n\pi}{p+1}\right) - \cos\left(\frac{2m\pi}{p}\right)}{\sin^3\left(\frac{n\pi}{p+1}\right) \sin^3\left(\frac{m\pi}{p}\right)}}\, \exp\left(-\frac{1}{g}\Gamma_{ba}(z_*)\right) \tag{3.29}$$

with

$$\Gamma_{ba}(z_*) = -\Gamma_{ab}(z_*)$$

$$= +\frac{8p}{2^{1/p}(2p+1)} \mu^{(2p+1)/2p} \sin\left(\frac{n}{p+1}\pi\right) \sin\left(\frac{m}{p}\pi\right) \quad (> 0). \tag{3.30}$$

Note that the expression (3.29) is invariant under the change of (m, n) into $(p-m-1, p-n)$. Thus we can always restrict the values of (m, n) to the region

$$0 < m < p - 1, \quad 0 < n < p, \quad m(p+1) - np > 0. \tag{3.31}$$

For example, in the pure gravity case $(p = 2)$, the only possible choice is $(m, n) = (1, 1)$ or $(a, b; l) = (0, 1; 0)$, for which we obtain

$$\langle D_{01}\rangle = \frac{2^{1/4}\, g^{1/2}}{8\, \pi^{1/2}\, 3^{3/4}} \mu^{-5/8} \exp\left[-\frac{4\sqrt{6}}{5g} \mu^{5/4}\right]. \tag{3.32}$$

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By rescaling the string coupling constant as $g = g_s/\sqrt{2}$, it becomes[9]

$$\langle D_{01} \rangle = \frac{g_s^{1/2}}{8\,\pi^{1/2}\,3^{3/4}}\,\mu^{-5/8}\,\exp\left[-\frac{8\sqrt{3}}{5g_s}\,\mu^{5/4}\right]. \tag{3.33}$$

This coincides, up to a factor of i, with the partition function of a D-instanton evaluated by resorting to one-matrix model [19] (see also [13, 14]). We shall make a comment on this discrepancy in section 6.

4   Comparison with Liouville field theory

In our analysis made in the preceding sections, the operators which are physically meaningful are the macroscopic loop operator $\varphi_0(\zeta)$ and the soliton operators $D_{ab} = \oint d\zeta\,\bar{c}_a(\zeta)\,c_b(\zeta)$ $(a \neq b)$. They should have their own correspondents in Liouville field theory. The former evidently corresponds to FZZT branes. For example, with the parametrization (3.8) and (3.9), the annulus amplitude of FZZT branes (eq. (2.17)) is expressed (up to nonuniversal additive constants) as

$$\left\langle \varphi_0(\zeta_1)\,\varphi_0(\zeta_2) \right\rangle_c^{(0)} = \ln \frac{z_1 - z_2}{T_p(z_1) - T_p(z_2)}, \tag{4.1}$$

and agrees with the calculation based on Liouville field theory [6, 15].

On the other hand, the relation between the soliton operators and ZZ branes is indirect. In fact, our solitons can take arbitrary positions for finite values of g, but in the weak coupling limit they get localized at saddle points and become ZZ branes. In this section we shall establish this relationship between the localized solitons and the ZZ branes with explicit correspondence between their quantum numbers.

A detailed analysis of FZZT and ZZ branes in (p, q) minimal string theory is performed in [11]. According to this, the BRST equivalence classes of ZZ branes are labeled by two quantum numbers (m, n), and their boundary states can be written as differences of two FZZT boundary states:

$$\left| m, n \right\rangle_{\text{ZZ}} = \left| \zeta(z_{mn}^+) \right\rangle_{\text{FZZT}} - \left| \zeta(z_{mn}^-) \right\rangle_{\text{FZZT}}. \tag{4.2}$$

Here $\zeta(z) = \sqrt{\mu}\,T_p(z)$ denotes the boundary cosmological constant of an FZZT brane. $z_{mn}^\pm$

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are the singular points of the Riemann surface $\mathcal{M}_{p,q}$ which z uniformizes, and are given by

$$z_{mn}^{\pm} = \cos \frac{\pi(mq \pm np)}{pq}, \tag{4.3}$$

$$m = 1, \cdots, p-1, \quad n = 1, \cdots, q-1, \quad mq - np > 0. \tag{4.4}$$

From this, one obtains the relation

$$Z_{\mathrm{ZZ}}^{(m,n)} = Z_{\mathrm{FZZT}}\big(\zeta(z_{mn}^{+})\big) - Z_{\mathrm{FZZT}}\big(\zeta(z_{mn}^{-})\big), \tag{4.5}$$

where $Z_{\mathrm{ZZ}}^{(m,n)}$ is the disk amplitude of a ZZ brane with quantum number (m,n), and $Z_{\mathrm{FZZT}}(\zeta)$ is that of an FZZT brane with boundary cosmological constant ζ.

On the other hand, our analysis shows that in the weak coupling limit $g \to +0$, the partition function in the presence of a soliton, $\langle D_{ab}\rangle$, is dominated by a saddle point z_* of the function $\Gamma_{ab}(z) = \langle \phi_a(z)\rangle^{(0)} - \langle \phi_b(z)\rangle^{(0)}$, and is expressed as $\langle D_{ab}\rangle \sim e^{(1/g)\Gamma_{ab}(z_*)}$. This implies that $\Gamma_{ab}(z_*)$ (< 0) should be regarded as the disk amplitude of a D-instanton [20]. Furthermore, it was explicitly evaluated at the saddle point $z = z_*(a,b;l) = \cos\big(-\frac{a+b}{p} + \frac{a+b+l}{p+1}\big)\pi$ in the previous section as

$$\begin{aligned}
\Gamma_{ab}(z_*) &= \big\langle \phi_a(z_*)\big\rangle^{(0)} - \big\langle \phi_b(z_*)\big\rangle^{(0)} \\
&= \Big\langle \phi_0\big(z_{*a}(a,b;l)\big)\Big\rangle^{(0)} - \Big\langle \phi_0\big(z_{*b}(a,b;l)\big)\Big\rangle^{(0)} \\
&= \Big\langle \varphi_0\big(\zeta(z_{*a}(a,b;l))\big)\Big\rangle^{(0)} - \Big\langle \varphi_0\big(\zeta(z_{*b}(a,b;l))\big)\Big\rangle^{(0)} \\
&= Z_{\mathrm{FZZT}}\big(\zeta(z_{*a}(a,b;l))\big) - Z_{\mathrm{FZZT}}\big(\zeta(z_{*b}(a,b;l))\big). \tag{4.6}
\end{aligned}$$

Here z_{*a} and z_{*b} are calculated in (3.20) and (3.21) as

$$z_{*a}(a,b;l) = \cos\left(\frac{m}{p} - \frac{n}{p+1}\right)\pi, \quad z_{*b}(a,b;l) = \cos\left(\frac{m}{p} + \frac{n}{p+1}\right)\pi, \tag{4.7}$$

with $(a,b;l)$ being related to (m,n) as in (3.22). We thus see that the shifted points $z_{*a}(a,b;l)$ and $z_{*b}(a,b;l)$ correspond to the singular points $z_{mn}^{\pm}$ as

$$z_{*a}(a,b;l) = z_{mn}^{-}, \quad z_{*b}(a,b;l) = z_{mn}^{+}. \tag{4.8}$$

Then (4.5), (4.6) and (4.8) lead to the following equality between $\Gamma_{ab}(z_*)$ and $Z_{\mathrm{ZZ}}^{(m,n)}$:

$$\Gamma_{ab}(z_*) = \big\langle \phi_0(z_{mn}^{-})\big\rangle - \big\langle \phi_0(z_{mn}^{+})\big\rangle = -Z_{\mathrm{ZZ}}^{(m=b-a,\, n=a+b+l)}. \tag{4.9}$$

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Therefore, each saddle point corresponds to a ZZ brane.[10] The relative minus sign between $\Gamma_{ab}(z_*)$ (taken to be negative for the convergence in the weak coupling limit) and $Z_{\mathrm{ZZ}}^{(m,n)}$ (conventionally normalized to be positive) appearing in (4.9) is naturally derived in our analysis and matched with the argument given in [15].

For the rest of this section, we illustrate the above correspondence along the line of the geometric setting introduced in [11]. We only consider the pure gravity case ($p = 2$), but the generalization to other cases must be straightforward.

For $p = 2$, the amplitude of a ZZ brane is given by Γ_{10}. Furthermore, using the W_n^k constraint with $k = 1$, one can easily show that $\langle \partial \varphi_1(\zeta) \rangle = -\langle \partial \varphi_0(\zeta) \rangle$. Thus the saddle points of Γ_{10} can be determined simply by solving the following equation:

$$0 = \partial_\zeta \Gamma_{01}(\zeta) = 2\langle \partial \varphi_0(\zeta) \rangle = \frac{8}{3}\sqrt{f_3(\zeta)}, \qquad f_3(\zeta) \equiv \left(\zeta - \frac{\sqrt{\mu}}{2} \right)^2 (\zeta + \sqrt{\mu}). \qquad (4.10)$$

Here $f_3(\zeta)$ is a degree-three polynomial of ζ, and has a single root at $\zeta = -\sqrt{\mu}$ and a double root at $\zeta = \sqrt{\mu}/2$. The algebraic curve defined by $y^2 = f_3(\zeta)$ is thus a torus with a pinched cycle, as depicted in Fig. 1.
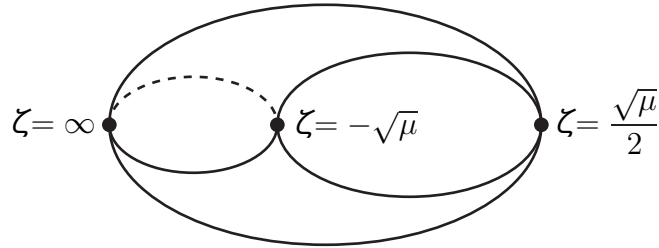

Fig. 1. Pinched Riemann surface

Using the relation $\zeta/\sqrt{\mu} = T_2(z) = 2z^2 - 1$, these roots are expressed in terms of z as

$$\zeta_* = -\sqrt{\mu} \Longleftrightarrow z_* = 0, \qquad \zeta_* = \frac{\sqrt{\mu}}{2} \Longleftrightarrow z_* = \pm\frac{\sqrt{3}}{2}. \qquad (4.11)$$

The first saddle point $z_* = 0$ corresponds to the zero of $U_{p-1}(z) = U_1(z) = 2z$ which was discarded in our analysis because such saddle points give rise to vanishing measure. We thus see that the pinched cycle corresponds to the D-instanton and thus to the ZZ brane.

---

[10]We should stress that it is not the saddle point z_* (the "position" of the D instanton) but the shifted points z_{*a} and z_{*b} which actually correspond to the singular points in the Riemann surface $\mathcal{M}_{p,p+1}$, although the set of the saddle points coincides with that of the singular points.

If we take $(a, b; l) = (0, 1; 0)$ as before, this selects the saddle point at $z_*(0, 1; 0) = \sqrt{3}/2$. Then its shifted points are calculated as

$$z_{*0} = \frac{\sqrt{3}}{2} = z_{11}^-, \qquad z_{*1} = -\frac{\sqrt{3}}{2} = z_{11}^+. \tag{4.12}$$

5 Annulus amplitudes of D-instantons

The annulus amplitudes of distinct D-instantons (ZZ branes) can also be calculated easily. These amplitudes correspond to the states

$$D_{ab} D_{cd} |\Phi\rangle, \tag{5.1}$$

which appear, for example, when two distinct solitons are present in the background: $e^{D_{ab} + D_{cd}} |\Phi\rangle$.

The two-point function of two solitons, $\langle D_{ab} D_{cd} \rangle$, can be written as

$$
\begin{aligned}
\langle D_{ab} D_{cd} \rangle &= \oint d\zeta \oint d\zeta' \frac{\langle -B/g \,|\, :e^{\varphi_a(\zeta) - \varphi_b(\zeta)} :\, :e^{\varphi_c(\zeta') - \varphi_d(\zeta')} :\, |\Phi\rangle}{\langle -B/g \,|\, \Phi\rangle} \\
&= \oint d\zeta \oint d\zeta' \, e^{(\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}) \ln(\zeta - \zeta')} \left\langle e^{\varphi_a(\zeta) - \varphi_b(\zeta) + \varphi_c(\zeta') - \varphi_d(\zeta')} \right\rangle \\
&= \oint d\zeta \oint d\zeta' \, e^{(\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}) \ln(\zeta - \zeta')} \, \exp \left\langle e^{\varphi_a(\zeta) - \varphi_b(\zeta) + \varphi_c(\zeta') - \varphi_d(\zeta')} - 1 \right\rangle_c \\
&= \oint d\zeta \oint d\zeta' \, e^{(1/g)\Gamma_{ab}(\zeta) + (1/g)\Gamma_{cd}(\zeta')} \, e^{(1/2)K_{ab}(\zeta)} \, e^{(1/2)K_{cd}(\zeta')}. \\
&\quad \cdot e^{(\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}) \ln(\zeta - \zeta')} \, e^{\langle (\varphi_a(\zeta) - \varphi_b(\zeta))(\varphi_c(\zeta') - \varphi_d(\zeta')) \rangle_c^{(0)}} \, e^{O(g)}.
\end{aligned} \tag{5.2}
$$

Since D_{ab} and D_{cd} may have their own saddle points ζ_* and ζ_*' in the weak coupling limit, the two-point function will take the following form:

$$
\begin{aligned}
&\langle D_{ab} D_{cd} \rangle \\
&= \langle D_{ab} \rangle \langle D_{cd} \rangle \cdot \\
&\quad \cdot \exp \left[(\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}) \ln(\zeta_* - \zeta_*') + \langle (\varphi_a(\zeta_*) - \varphi_b(\zeta_*))(\varphi_c(\zeta_*') - \varphi_d(\zeta_*')) \rangle_c^{(0)} \right]. \tag{5.3}
\end{aligned}
$$

We thus identify the annulus amplitude of D-instantons as

$$
\begin{aligned}
K_{ab|cd}(z_*, z_*') &= \langle (\varphi_a(\zeta_*) - \varphi_b(\zeta_*))(\varphi_c(\zeta_*') - \varphi_d(\zeta_*')) \rangle_c^{(0)} + (\delta_{ac} + \delta_{bd} - \delta_{ad} - \delta_{bc}) \ln(\zeta_* - \zeta_*') \\
&= \langle (\phi_a(z_*) - \phi_b(z_*))(\phi_c(z_*') - \phi_d(z_*')) \rangle_c^{(0)} \\
&\quad + (\delta_{ac} + \delta_{cd} - \delta_{ad} - \delta_{bc}) \ln \left[\sqrt{\mu} \, (T_p(z_*) - T_p(z_*')) \right]. \tag{5.4}
\end{aligned}
$$

The right-hand side can be simplified by using (3.17), and we obtain

$$K_{ab|cd}(z_*, z_*') = \ln \frac{(z_{*a} - z_{*c}')(z_{*b} - z_{*d}')}{(z_{*a} - z_{*d}')(z_{*b} - z_{*c}')}$$

$$= \ln \frac{(z_{mn}^- - z_{m'n'}^-)(z_{mn}^+ - z_{m'n'}^+)}{(z_{mn}^- - z_{m'n'}^+)(z_{mn}^+ - z_{m'n'}^-)}$$

$$= Z_{\text{annulus}}^{(m,n|m',n')} \tag{5.5}$$

where we have used the identification (see (4.8))

$$z_{*a} = z_{mn}^-, \quad z_{*b} = z_{mn}^+, \tag{5.6}$$

$$z_{*c}' = z_{m'n'}^-, \quad z_{*d}' = z_{m'n'}^+. \tag{5.7}$$

This expression correctly reproduces the annulus amplitudes of ZZ branes obtained in [6, 15].

6   Conclusion and discussions

In this paper, we have studied D-instantons of unitary $(p, p+1)$ minimal strings in terms of the continuum string field theory [1]. In this formulation, there are two types of operators that have definite physical meanings; One is the (unmarked) macroscopic loop operator $\varphi_0(\zeta)$, and the other is of the soliton operators D_{ab}. We have calculated the expectation value of the soliton operator $\langle D_{ab} \rangle$ in the weak coupling limit $g \to 0$. $\langle D_{ab} \rangle$ is then expanded as $\langle D_{ab} \rangle = \int d\zeta \, \exp[(1/g)\Gamma_{ab} + (1/2)K_{ab} + O(g)]$, and is dominated by saddle points. We have carefully identified the valid saddle points, and found that $\langle D_{ab} \rangle$ has a well-defined finite value. Since there is no ambiguity in obtaining it, this expectation value must be universal.

Each saddle point denoted by $z_*(a, b; l)$ corresponds to the location of a localized soliton. On the other hand, ZZ branes in Liouville field theory are characterized by a pair of FZZT cosmological constants $z_{mn}^\pm$ [6], which correspond to singular points on a Riemann surface $\mathcal{M}_{p,q}$ [11]. We have shown that the free energy of the localized soliton, $\Gamma_{ab}(z_*)$, can be identified with minus the partition function of the ZZ brane $Z_{\text{ZZ}}^{(m,n)}$ with the following relation:

$$b - a = m, \quad a + b + l = n,$$

$$z_{*a}(a, b; l) = z_{mn}^-, \quad z_{*b}(a, b; l) = z_{mn}^+. \tag{6.1}$$

This identification is actually valid for any (p, q) minimal string theories. We pointed out that the set $\{(a, b; l)\}$ are redundant and can be restricted to a smaller set with $1 \leq b - a\,(= m) \leq p - 1$, $1 \leq a + b + l\,(= n) \leq q - 1$ and $mq - np > 0$.

With the identification (6.1) we have shown that the annulus amplitudes of localized solitons, $K_{ab|cd}(z_*, z'_*)$, coincide with those of ZZ branes, $Z_{\text{annulus}}^{(m,n|m',n')}$ [6, 15]. There is no ambiguity in deriving this equivalence, we thus can conclude that these annulus amplitudes also must be universal and can be derived from the saddle point analysis.

For the pure gravity case $(p = 2)$ one can compare the resulting value $\langle D_{ab} \rangle$ with the one-instanton partition function obtained with the use of matrix models [19, 13, 14]. We found that they coincide up to a single factor i. In fact, the overall normalization of D_{ab} is not fixed only from the continuum loop equations. However, if D_{ab} creates a soliton with a definite charge, then D_{ba} creates an anti-soliton with the opposite charge. Moreover, with the present normalization of D_{ab}, the operator $D_{ab} D_{ba}$ almost gives identity for the 0-soliton sector, so that the present normalization seems to be natural. In this sense, this discrepancy between our result and the matrix model result deserves explanation. Among possible ones are (i) that it may be natural to introduce i in defining the fugacity, or (ii) that the D-instanton calculation in matrix models may choose a path different from the steepest-descent one for Γ_{ab}.

In this paper, we mainly consider the unitary $(p, p + 1)$ minimal strings. The analysis can be easily extended to general (p, q) minimal strings, as will be reported in our future communication.

Acknowledgments

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